stichting
mathematisch
centrum

MC

AFDELING ZUIVERE WISKUNDE ZW 109/78 FEBRUARI
(DEPARTMENT OF PURE MATHEMATICS)

A.E. BROUWER, A.J. DE VRIES & R.M.A. WIERINGA

A LOWER BOUND FOR THE LENGTH OF PARTIAL TRANSVERSALS IN A LATIN SQUARE

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 05B15

A lower bound for the length of partial transversals in a Latin square

by

A.E. Brouwer, A.J. de Vries*) & R.M.A. Wieringa*)

ABSTRACT

Any Latin square of order n has a partial transversal of order at least $n - \sqrt{n}$.

KEY WORDS & PHRASES: *latin square, transversal.*

---

*) Eindhoven University of Technology

This report will be submitted for publication elsewhere.

0. INTRODUCTION

Let A be a Latin square of order n (i.e. a square matrix such that each of its rows and columns is a permutation of the set $I_n = \{1,2,\ldots,n\}$). A set $T \subset I_n \times I_n$ is called a partial transversal of A if $|T| = \#\{i \mid (i,j) \in T\} = \#\{j \mid (i,j) \in T\} = \#\{a(i,j) \mid (i,j) \in T\}$ (i.e. no two positions in the same row or column; no two entries the same).

In [3] KOKSMA proved that for $n \geq 3$ each latin square of order n has a partial transversal of length at least $(2n+1)/3$. A simple modification of his method yields a lower bound of $(3n-1)/4$. DRAKE [2] proved the existence of a partial transversal of length at least $3n/4$ for $n \geq 8$. DE VRIES & WIERINGA [4] perfected Koksma's method and were able to prove a lower bound of $(4n-3)/5$ for $n \geq 12$. Here we prove a lower bound roughly of order $n - \sqrt{n}$. (It is sharper than the older bounds for $n \geq 7$.) This result is still far from the best possible; in fact RYSER (see [3]) and BRUALDI (see [1], p. 103) conjectured that any Latin square of order n has a partial transversal of order n-1.

For even n this result would be best possible since a circulant Latin square of even order does not possess a transversal, but it is probably true that each latin square of odd order has a transversal.

1. A LOWER BOUND

THEOREM. *Every latin square of order* n *has a partial transversal of order* $n-r$ *for an* r *with* $r(r+1) \leq n$.

PROOF. Let the longest partial transversal of a given Latin square A have length t and let $r = n-t$. By permuting rows, columns and symbols (if necessary) we may assume that $a(i,i) = i$ for $1 \leq i \leq t$. Let $L = \{t+1,\ldots,n\}$, the set of 'large' numbers.

Define sets A_i $(0 \leq i \leq r)$ by induction on i:

$$A_0 = \emptyset,$$

$$A_i = \{j \mid a(j,t+i) \in A_{i-1} \cup L\}.$$

Define a directed graph G with vertex set $\bigcup_{i=1}^{r} A_i \times \{t+i\}$ and edge set

$$\{((x,t+i),(y,t+j)) \mid i < j \text{ and } x = a(y,t+j)\}.$$

<u>LEMMA</u>. G *does not contain a directed path starting in a position* $(g,t+i)$ *where* A *has a large entry, and ending in a position* $(h,t+j)$ *with a large row number (i.e.* $h \in L$*).*

<u>PROOF</u>. Suppose $(g_0,t+i_0),(g_1,t+i_1),\dots,(g_\ell,t+i_\ell)$ is the shortest such path. Then the collection of $t+1$ positions

$$(g_k,t+i_k) \qquad k = 0,\dots,\ell$$

and

$$(j,j) \qquad \text{for } j \neq g_k \ (0 \le k \le \ell-1),\ j \le t$$

is a partial transversal, contradicting the definition of t.
For:

(i) Since $1 \le i_0 < i_1 < \dots < i_\ell$ all positions are in different columns.

(ii) All positions are in different rows since $j \neq g_k$, and if $g_h = g_k$ for $h < k$ then *either* $k = \ell$, and $g_h = g_k \in L$, so that

$$(g_0,t+i_0),\dots,(g_h,t+i_h)$$

is a shorter path down, *or* $k < \ell$ and $g_h = g_k = a(g_{k+1},t+i_{k+1})$ so that

$$(g_0,t+i_0),\dots,(g_h,t+i_h),(g_{k+1},t+i_{k+1}),\dots,(g_\ell,t+i_\ell)$$

is a shorter path down. Contradiction in both cases.

(iii) All entries are different, for $a(g_{k+1},t+i_{k+1}) = g_k$ so that the entries are the numbers $1,\dots,t$ and $a(g_0,t+i_0)$, where the latter is in L. □

From the lemma it follows that all vertices of G are in rows $1,\dots,t$.

It is also clear that $|A_i| = |A_{i-1}| + r$. Hence $|A_r| = r^2$. Therefore $r^2 \leq t$, i.e. $r^2 + r \leq n$. $\square$

REMARKS.

(1) For this presentation of our proof we are indebted to A. Schrijver.

(2) From an observation by J.H. van Lint it follows that equality can occur only for $n = 2$, i.e. for $n \geq 3$ we have $n \geq r(r+1) + 1$.

REFERENCES

[1] DÉNES, J. & A.D. KEEDWELL, *Latin squares and their applications*, English Universities Press, London, 1974.

[2] DRAKE, D.A., *Maximal sets of Latin squares and partial transversals*, J. of Statist. Planning and Inference 1 (1977) 143-149.

[3] KOKSMA, K.K., *A lower bound for the order of a partial transversal in a Latin square*, J. Combinatorial Theory 7 (1969) 94-95.

[4] VRIES, A.J. de & R.M.A. WIERINGA, *Een ondergrens voor de lengte van een partiële transversaal in een Latijns vierkant*, THE Memorandum 78-02.

MC, 780216

Added in proof

Essentially the same results were obtained by D.E. WOOLBRIGHT [5].

[5] WOOLBRIGHT, D.E., *An* n×n *Latin square has a transversal with at least* $n-\sqrt{n}$ *distinct symbols*, J. Combinatorial Theory (A) 24 (1978) 235-237.